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INVOLUTORIC CIRCULAR TRANSFORMATIONS AS A PARTICULAR CASE OF THE STEINERIAN TRANSFORMATION AND THEIR INVARIANT NETS OF CUBICS.*

By Arnold Emch.

It is well known that with every Steinerian † or involutoric quadratic transformation in a plane is associated an invariant net of cubics passing through the fundamental quadruple of the transformation and its diagonal points. Conversely, every plane cubic admits of an infinite number of birational transformations in itself, which form a continuous group. In case of a rational cubic (deficiency 0) the group depends upon three parameters. For a cubic of deficiency 1, there exists a mixed one-parameter group consisting of a linear system g_2^1 of Steinerian transformations, and a group of non-central transformations. The transformations of the first kind do not form a group and are involutoric, while those of the second kind form a one-parameter group with three involutoric transformations. The product of two transformations of the first kind is equivalent with a transformation of the second kind.‡

The involutoric circular transformation in a complex plane

$$z' = \frac{az + b}{cz - a}$$

is a particular case of a Steinerian transformation. It is the purpose of this paper to establish this equivalence and the principal properties of the two nets of cubics which remain invariant in the transformations of the first and second kind, respectively.

In order to gain a general standpoint, I shall first briefly discuss the well-known conditions for conformal rational transformations in a plane, and then apply the results to birational quadratic transformations. This will lead immediately to general circular—and in particular to circular transformations in the Steinerian form.§

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* Read before the American Mathematical Society in Chicago, Dec. 29, 1911.

† Steiner, Werke, vol. I, pp. 409–421; Disteli, "Die Metrik der circularen Curven dritter Ordnung," Vierteljahrsschrift Zürich, vol. 37, pp. 255–305.

‡ Weyer, "Über eindeutige Beziehungen auf einer allgemeinen ebenen Curve dritter Ordnung," Wiener Ber., vol. 87, pp. 837–872; Segre, "Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes," Math. Ann., vol. 27, p. 296.

§ A discussion of the five Newtonian types of plane cubics and their construction by means of the Steinerian transformation was given by the author in the Univ. of Colorado Studies, vol. 1, pp. 275–284.

1. General Rational Algebraic Transformations.

Let R_1, R_2, R_3, R_4 designate polynomials in x and y of any degree, then

$$(1) \qquad x' = \frac{R_1}{R_3}, \qquad y' = \frac{R_2}{R_4}$$

represents a general rational algebraic transformation of the xy-plane into the $x'y'$-plane. It is furthermore assumed that the fractions R_1/R_3 and R_2/R_4 are irreducible. To every point (x, y) corresponds uniformly a point (x', y'). Conversely, to a point (x', y') correspond the points of intersection of the curves $x' R_3 - R_1 = 0$ and $y' R_4 - R_2 = 0$. Thus, in case of a general quadratic transformation, there is a (1, 4) correspondence between the points of the two planes. Applying the conditions for conformity

$$(2) \qquad \frac{\partial x'}{\partial x} = \frac{\partial y'}{\partial y}, \qquad \frac{\partial x'}{\partial y} = -\frac{\partial y'}{\partial x}$$

to (1), we get

$$(3) \qquad \frac{\dfrac{\partial R_1}{\partial x} R_3 - \dfrac{\partial R_3}{\partial x} R_1}{R_3^2} = \frac{\dfrac{\partial R_2}{\partial y} R_4 - \dfrac{\partial R_4}{\partial y} R_2}{R_4^2},$$

$$(4) \qquad \frac{\dfrac{\partial R_1}{\partial y} R_3 - \dfrac{\partial R_3}{\partial y} R_1}{R_3^2} = \frac{\dfrac{\partial R_2}{\partial x} R_4 - \dfrac{\partial R_4}{\partial x} R_2}{R_4^2}.$$

(3) and (4) must be satisfied for all possible pairs of values of x and y. But two rational fractions in x and y which are equal for all pairs of values of x and y have identical numerators and denominators (except as to a factor of proportionality which we may assume as cancelled). Hence $R_4 = R_3$. The other possibility $R_4 = -R_3$ may be disregarded, since it may be obtained from the first merely by a reflection on the x-axis. Thus putting in (3) and (4) $R_4 = R_3$, from these equations follows

$$\left(\frac{\partial R_1}{\partial x} R_3 - \frac{\partial R_3}{\partial x} R_1\right)\left(\frac{\partial R_2}{\partial x} R_3 - \frac{\partial R_3}{\partial x} R_2\right) + \left(\frac{\partial R_1}{\partial y} R_3 - \frac{\partial R_3}{\partial y} R_1\right)\left(\frac{\partial R_2}{\partial y} R_3 - \frac{\partial R_3}{\partial y} R_2\right) = 0,$$

or, dividing through by R_3, and replacing R_1/R_3 by x' and R_2/R_3 by y', and developing:

$$(5) \quad x'y'\left[\left(\frac{\partial R_3}{\partial x}\right)^2 + \left(\frac{\partial R_3}{\partial y}\right)^2\right] - x'\left(\frac{\partial R_3}{\partial x} \cdot \frac{\partial R_2}{\partial x} + \frac{\partial R_3}{\partial y} \cdot \frac{\partial R_2}{\partial y}\right) - y'\left(\frac{\partial R_1}{\partial x} \cdot \frac{\partial R_3}{\partial x} + \frac{\partial R_1}{\partial y} \cdot \frac{\partial R_3}{\partial y}\right) + \left(\frac{\partial R_1}{\partial x} \cdot \frac{\partial R_2}{\partial x} + \frac{\partial R_1}{\partial y} \cdot \frac{\partial R_2}{\partial y}\right) = 0.$$

This, as can easily be established, is the condition that all curves of the two pencils $x'R_3 - R_1 = 0$ and $y'R_3 - R_2 = 0$ are orthogonal to each other, and in case of conformity is a well-known fact. From (5) we conclude that also *the curves* $R_1 = 0$, $R_2 = 0$ *and* $R_3 = 0$ *are orthogonal among themselves.* To the infinite corresponds the curve $R_3 = 0$, which according to (5), when $x' = \infty$, $y' = \infty$, must satisfy the differential equation

$$\left(\frac{\partial R_3}{\partial x}\right)^2 + \left(\frac{\partial R_3}{\partial y}\right)^2 = 0. \tag{6}$$

The general solution of this equation is of the form

$$[\phi(x, y)]^2 + [\psi(x, y)]^2 = 0, \tag{7}$$

in which both $\phi\,(x, y)$ and $\psi\,(x, y)$ satisfy Laplace's differential equation.* In case of a rational transformation ϕ and ψ are polynomials in x and y. The real part of the curve $R_3 = 0$ consists therefore only of a limited number of points, namely the real points of intersection of the two curves

$$\phi(x, y) = 0 \qquad \text{and} \qquad \psi(x, y) = 0.$$

As $R_1 = 0$ and $R_2 = 0$ are orthogonal to $R_3 = 0$, they necessarily pass through these points of intersection. By means of the foregoing properties it is now easy to establish conformal quadratic transformations.

2. Conformal Quadratic Transformations.

In this case the pencils $x'R_3 - R_1 = 0$ and $y'R_3 - R_2 = 0$ must form two orthogonal pencils of conics. This is possible only when the conics are circles, so that the two pencils are conjugate. But as R_3 is common to both pencils, the real part of the circle $R_3 = 0$ must necessarily degenerate into a point. The two pencils have therefore a common point and any curve of the first pencil cuts every curve of the second pencil in one point only. From this it follows that the transformation is birational and of the form†

$$\left\{\begin{aligned} x' &= \frac{A_2B_3 - A_3B_2}{A_1B_2 - A_2B_1}, \\ y' &= \frac{A_3B_1 - A_1B_3}{A_1B_2 - A_2B_1}, \end{aligned}\right. \tag{8}$$

where the A's and B's represent arbitrary linear expressions in x and y. According to (2) and (7) $R_3 = A_1B_2 - A_2B_1$ has the form

$$(c_1x - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2,$$

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* There is of course: $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$, $\dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$.

† K. Doehlemann: Geometrische Transformationen, vol. II, pp. 56–78.

hence

$$-B_1 = A_2 = c_1x - c_2y + d_1,$$

and

$$A_1 = B_2 = c_1y + c_2x + d_2.$$

Substituting these expressions in (8) and replacing A_3 and B_3 also by linear expressions the transformation assumes the form

$$x' = \frac{(\alpha_1x + \alpha_2y + b_1)(c_1x - c_2y + d_1) + (a_1y + a_2x + b_2)(c_1y + c_2x + d_2)}{(c_1x - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2},$$

$$y' = \frac{(a_1y + a_2x + b_2)\,(c_1x - c_2y + d_1) - (\alpha_1x + \alpha_2y + b_1)\,(c_1y + c_2x + d_2)}{(c_1x_1 - c_2y + d_1)^2 + (c_1y + c_2x + d_2)^2}.$$

It will be noticed that so far the coefficients α_1, α_2, b_1, $-a_1$, $-a_2$, $-b_2$ are entirely arbitrary and independent of c_1, c_2, d_2. The numerators of these expressions however must also represent circles, which requires the equality of the coefficients of x^2 and y^2 in each case and the vanishing of the coefficients of xy. This leads to the two conditions

$$\alpha_1c_1 + \alpha_2c_2 = a_1c_1 - a_2c_2,$$

$$\alpha_1c_2 - \alpha_2c_1 = a_1c_2 + a_2c_1.$$

Solving fot α_1 and α_2, we find $\alpha_1 = a_1$, $\alpha_2 = -a_2$. Writing $p_1 = a_1x - a_2y + b_1$, $q_1 = c_1x - c_2y + d_1$, $p_2 = a_1y + a_2x + b_2$, $q_2 = c_1y + c_2x + d_2$ the equations of the circles $R_1 = 0$ and $R_2 = 0$ may be written in the form

$$(9) \qquad R_1 \equiv p_1q_1 + p_2q_2 = 0, \qquad R_2 \equiv p_2q_1 - p_1q_2 = 0.$$

From this is seen that both circles pass through the points of intersection of the rays $p_1 = 0$ and $p_2 = 0$, and $q_1 = 0$ and $q_2 = 0$. The first circle (R_1) passes also through the intersection of $p_1 = 0$ and $q_2 = 0$, and also of $q_1 = 0$ and $p_2 = 0$. The second circle (R_2) passes through the vertices of the pairs $p_1 = 0$, $q_1 = 0$ and $p_2 = 0$, $q_2 = 0$, Fig. 1. The real portion of $R_3 = 0$ is clearly the point of intersection of $q_1 = 0$ and $q_2 = 0$. From the equations of the rays p_1, p_2, q_1, q_2 it is seen that $p_1 \perp p_2$ and $q_1 \perp q_2$. (In Fig. 1 the letters contained in the equations are used to designate the corresponding rays and circles.) Hence, p_1 and q_1 are altitudes in the triangle XYZ, and also ZM, cutting XY in Z_1, is an altitude and $\triangle MY_1Z \backsim \triangle XY_1Y$. These two triangles are brought into a homologous position by turning them through an angle of 90°, and as the circles R_1 and R_2 are also homologously related to the triangles, it follows that in their original position they are orthogonal. This as we have seen in the general case, is a necessary condition. The orthogonality of R_1 and R_2 might of course also be proved analytically without difficulty. Writing the transformations in the form

$$x' = \frac{p_1q_1 + p_2q_2}{q_1^2 + q_2^2}, \qquad y' = \frac{p_2q_1 - p_1q_2}{q_1^2 + q_2^2}, \tag{10}$$

and making use of the identities

$$\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial y} = 0, \quad \frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial x} = 0, \quad \frac{\partial q_1}{\partial x} - \frac{\partial q_2}{\partial y} = 0, \quad \frac{\partial q_1}{\partial y} + \frac{\partial q_2}{\partial x} = 0,$$

it is easily verified that (10) satisfies (2). *The transformation is therefore conformal.* Substituting in the equation $ux' + vy' + w = 0$ for x' and y' their expressions as given by (10), the equation

$$u(p_1q_1 + p_2q_2) + v(p_2q_1 - p_1q_2) + w(q_1^2 + q_2^2) = 0$$

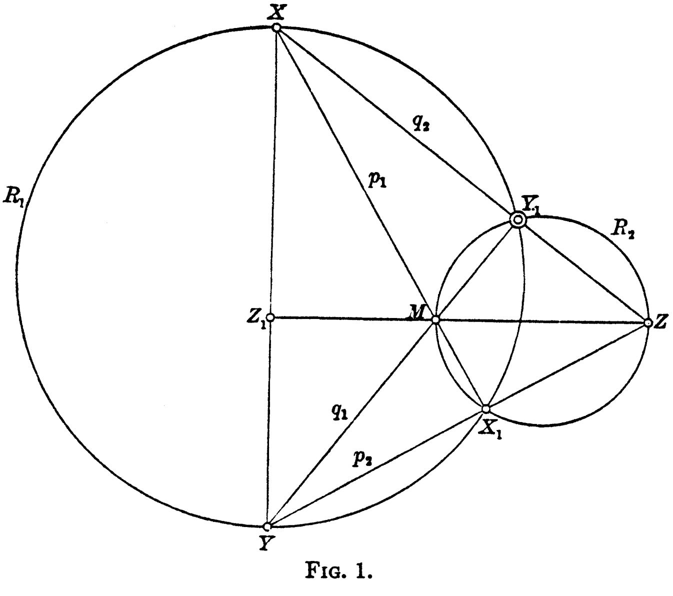

FIG. 1.

results. In each parenthesis the coefficients of x^2 and y^2 are equal, so that this equation represents a circle passing through Y_1, ($R_3 = 0$), the point corresponding to the infinite point of the straight line in the $x'y'$-plane. Consider now a point P' in the $x'y'$-plane and the pencil of rays through it. This pencil is transformed into the pencil of circles through Y_1 and P (corresponding to P') in the xy-plane. Any circle with P' as a center is transformed into a circle of the conjugate pencil in the xy-plane. Thus every circle in the $x'y'$-plane is transformed into a circle of the xy-plane. Conversely, every circle in the xy-plane may be considered as a circle of a pencil which is conjugate to a pencil having Y_1 as one of the base points. The two pencils are transformed into a pencil of rays and the corresponding systems of concentric circles. Every circle of the xy-plane is therefore transformed into a circle of the $x'y'$-plane. The transformation is therefore circular. By this the theorem is established:

All conformal quadratic transformations are circular transformations.

It remains to be shown that the transformation as represented by (14) may be expressed as a transformation between complex variables. For this purpose we have from (10)

$$x' + iy' = \frac{p_1q_1 - i \cdot ip_2q_2 + ip_2q_1 - ip_1q_2}{(q_1 + iq_2)(q_1 - iq_2)},$$

or

$$x' + iy' = \frac{(p_1 + ip_2)(q_1 - iq_2)}{(q_1 + iq_2)(q_1 - iq_2)},$$

or

$$x' + iy' = \frac{p_1 + ip_2}{q_1 + iq_2} = \frac{a_1x - a_2y + b_1 + i(a_1y + a_2x + b_2)}{c_1x - c_2y + d_1 + i(c_1y + c_2x + d_2)}.$$

Replacing $-a_2y$ by $ia_2 \cdot iy$ and $-c_2y$ by $ic_2 \cdot iy$, this, by factoring, becomes

$$x' + iy' = \frac{(a_1 + ia_2)(x + iy) + b_1 + ib_2}{(c_1 + ic_2)(x + iy) + d_1 + id_2},$$

or, in the usual form

$$z' = \frac{az + b}{cz + d}. \tag{11}$$

3. Involutoric Circular Transformation.

If in (11) $d = -a$, or $d_1 = -a_1$, $d_2 = -a_2$, the transformation becomes involutoric:

$$z' = \frac{az + b}{cz - a}, \tag{12}$$

or in Cartesian coördinates

$$(13)\quad \begin{cases} x' = \dfrac{(a_1x - a_2y + b_1)(c_1x - c_2y - a_1) + (a_1y + a_2x + b_2)(c_1y + c_2x - a_2)}{(c_1x - c_2y - a_1)^2 + (c_1y + c_2x - a_2)^2} \\[2ex] y' = \dfrac{(a_1y + a_2x + b_2)(c_1x - c_2y - a_1) - (a_1x - a_2y + b_1)(c_1y + c_2x - a_2)}{(c_1x - c_2y - a_1)^2 + (c_1y + c_2x - a_2)^2}. \end{cases}$$

The general involutoric quadratic (Steinerian) transformation of a plane in itself may be defined in the following manner:* Let

$$P_1 \equiv \alpha_1x^2 + 2\beta_1xy + \gamma_1y^2 + 2\delta_1x + 2\epsilon_1y + \varphi_1 = 0,$$

$$P_2 \equiv \alpha_2x^2 + 2\beta_2\,xy + \gamma_2y^2 + 2\delta_2x + 2\epsilon_2y + \varphi_2 = 0$$

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* Steiner, Disteli, loc. cit. R. Sturm: Die Lehre von den geometr. Verwandtschaften, vol. II, pp. 72–95; A. Emch: Introduction to Projective Geometry, pp. 185–216.

represent two distinct conics. Then the polars of a point (x', y') with respect to the pencil

$$P_1 + \lambda P_2 = 0$$

are concurrent at a point (x, y). Conversely all polars of the point (x, y) with respect to the pencil pass through the point (x', y'). Thus, to every point (x, y) corresponds a point (x', y') and conversely. The transformation established in this manner is involutoric and quadratic. It is evident that any two conics of the pencil, consequently also two of the degenerate conics may serve as a base of the transformation. Designate the quadrangle through which the pencil passes by $A_1A_2A_3A_4$ and its diagonal points by B_1, B_2, B_3. To find P' when P is given, join P to B_1, B_2, B_3, Fig. 2, and construct the fourth harmonic rays to PB_1, PB_2, PB_3 respectively

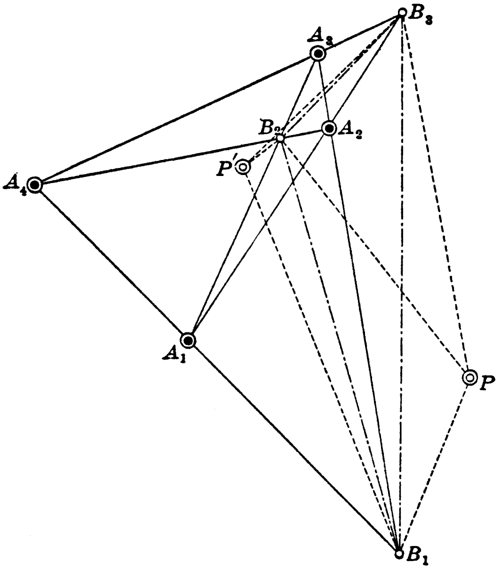

FIG. 2.

with respect to the corresponding pairs of lines (degenerate conics) through B_1, B_2, B_3. The three harmonic rays intersect each other at P'. To the points B correspond all points of their opposite sides of the triangle $B_1B_2B_3$. The points A_1, A_2, A_3, A_4 are invariant. To a point on any line joining any two A's, say A_1A_3, corresponds the fourth harmonic point to the pair A_1A_3 on the same line. All other points are in uniform correspondence. To the straight lines of the plane corresponds the net of conics through $B_1B_2B_3$. To the line at infinity corresponds a conic through these points which bisects the sides of the quadrilateral. This conic is a circle if two of the degenerate conics consist of rectangular pairs of lines. In this case the conics of the pencil are all equilateral hyperbolas. Conversely, two equilateral hyperbolas determine a pencil, whose degenerate conics consist of rectangular pairs of lines and all of whose conics are equilateral hyperbolas.

The analytic form of the general involutoric quadratic transformation is

$$(14)\quad \begin{cases} x' = \dfrac{(\beta_1 x + \gamma_1 y + \epsilon_1)(\delta_2 x + \epsilon_2 y + \varphi_2) - (\beta_2 x + \gamma_2 y + \epsilon_2)(\delta_1 x + \epsilon_1 y + \varphi_1)}{(\alpha_1 x + \beta_1 y + \delta_1)(\beta_2 x + \gamma_2 y + \epsilon_2) - (\alpha_2 x + \beta_2 y + \delta_2)(\beta_1 x + \gamma_1 y + \epsilon_1)}, \\ y' = \dfrac{(\alpha_2 x + \beta_2 y + \delta_2)(\delta_1 x + \epsilon_1 y + \varphi_1) - (\alpha_1 x + \beta_1 y + \delta_1)(\delta_2 x + \epsilon_2 y + \varphi_2)}{(\alpha_1 x + \beta_1 y + \delta_1)(\beta_2 x + \gamma_2 y + \epsilon_2) - (\alpha_2 x + \beta_2 y + \delta_2)(\beta_1 x + \gamma_1 y + \epsilon_1)}. \end{cases}$$

In order that this be at the same time a circular transformation, we must dispose of the coefficients in such a manner that (14) becomes identical with (13). This is the case, if we put

$$\begin{array}{llll} \alpha_1 = c_2, & \varphi_2 = b_1, & \alpha_2 = -c_1, & \varphi_1 = -b_2 \\ \beta_1 = c_1, & \gamma_2 = c_1, & \beta_2 = c_2, & \gamma_1 = -c_2 \\ \delta = d_2 = -a_2, & \epsilon_2 = d_2 = -a_2, & \delta_2 = -d_1 = a_1, & \epsilon_1 = d_1 = -a_1. \end{array}$$

The conics which determine the pencil are now

$$(15)\quad \begin{cases} P_1 \equiv c_2 x^2 + 2c_1 xy - c_2 y^2 - 2a_2 x - 2a_1 y - b_2 = 0, \\ P_2 \equiv -c_1 x^2 + 2c_2 xy + c_1 y^2 + 2a_1 x - 2a_2 y + b_1 = 0. \end{cases}$$

Eliminating from these equations in turn xy, and x^2 and y^2, the two new conics belonging to the same pencil arise:

$$(16)\quad \begin{cases} Q_1 \equiv x^2 - y^2 - 2\dfrac{a_1c_1 + a_2c_2}{c_1^2 + c_2^2}x + 2\dfrac{a_2c_1 - a_1c_2}{c_1^2 + c_2^2}y - \dfrac{b_1c_1 + b_2c_2}{c_1^2 + c_2^2} = 0, \\ Q_2 \equiv 2xy - 2\dfrac{a_2c_1 - a_1c_2}{c_1^2 + c_2^2}x - 2\dfrac{a_1c_1 + a_2c_2}{c_1^2 + c_2^2}y + \dfrac{b_1c_2 - b_2c_1}{c_1^2 + c_2^2} = 0. \end{cases}$$

They clearly represent two equilateral hyperbolas. All conics of the pencil are therefore equilateral hyperbolas and the degenerate conics are three pairs of rectangular lines with the points B_1, B_2, B_3 as vertices. To the line at infinity corresponds a circle through these three points, whose equation is

$$(17)\quad R_3 \equiv (c_1 x - c_2 y - a_1)^2 + (c_1 y + c_2 x - a_2)^2 = 0.$$

The real portion of this circle is the point Y_1 with the coördinates

$$(18)\quad s = \frac{a_1c_1 + a_2\,c_2}{c_1^2 + c_2^2}, \qquad t = \frac{a_2c_1 - a_1c_2,}{c_1^2 + c_2^2},$$

which according to (16) is apparently the common center of the two hyperbolas $Q_1 = 0$, $Q_2 = 0$. As the circles corresponding to the straight lines of the plane all pass through B_1, B_2, B_3, it follows that these points coincide with the point Y_1 ($R_3 = 0$) and the circular points at infinity. The quadrangle $A_1A_2A_3A_4$ has therefore only two real points, while the remaining two are conjugate imaginary. Writing $a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$, and forming $Q_1 + iQ_2 = 0$, and reducing we find

$$cz^2 - 2az - b = 0, \tag{19}$$

which is recognized as the equation for the double points of the transformation. The real points of intersection of the two hyperbolas coincide with these double-points and are therefore given by

$$A_1 \equiv \frac{a + \sqrt{a^2 + bc}}{c}, \qquad A_2 \equiv \frac{a - \sqrt{a^2 + bc}}{c}. \tag{20}$$

The middle-point B_3 between A_1 and A_2 is determined by a/c and in $z' = (az + b)/(cz - a)$ clearly corresponds to the point at infinity $z = \infty$. B_3 is the center of the involution and its coördinates, resulting by separating real and imaginary of a/c, are identical with s and t. Thus B_3 coincides with Y_1. The pencil of circles through A_1 and A_2 and its conjugate orthogonal pencil are invariant in the transformation. The two imaginary points of intersection of all circles of the conjugate pencil are identical with the imaginary pair A_3A_4 of the quadruple determined by the hyperbolas.* From the equations of these we see that any two equilateral orthogonal hyperbolas determine such a quadrangle. Hence, the theorem may be stated:

The Steinerian transformation (involutoric quadratic transformation) based on a pencil of conics determined by two equilateral orthogonal hyperbolas is identical with the involutoric circular transformation (involution) having the real pair of the quadruple of the pencil as double points.

4. The Invariant Cubics.

Designating by P and P' two corresponding points of a Steinerian transformation based on a quadruple $A_1A_2A_3A_4$ with the diagonal points $B_1B_2B_3$, every cubic through these nine points is left invariant by the transformation. With every point P in a general position is associated such a cubic, so that there is a net of cubics through the seven points (A) and (B), in which every cubic is invariant. Geometrically nothing is lost in generality if we assume the center of involution B_3 as the origin and the line joining A_1A_2 as the real axis. We may also assume $B_3A_1 = 1$, so that the circular transformation assumes the simple form

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* Darboux in "Sur une classe remarquable de courbes et de surfaces algébriques," pp. 61–66, calls A_3A_4 "points associés" of A_1A_2, and conversely.—Study in "Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft, pp. 8–19, calls A_1A_2 the "first picture" (erstes Bild) of A_3A_4.

$$z' = \frac{1}{z}. \tag{21}$$

The equations of the hyperbolas determining the pencil of conics through the fundamental quadruple are now

$$Q_1 \equiv x^2 - y^2 - 1 = 0, \qquad Q_2 \equiv 2xy = 0. \tag{22}$$

The polars of a point $P(x_1, y_1)$ with respect to these conics have the equations

$$q_1 \equiv x_1x - y_1y - 1 = 0, \qquad q_2 \equiv x_1y + y_1x = 0.$$

The two pencils $Q_1 + \lambda Q_2 = 0$ and $q_1 + \lambda q_2$ are projective and produce the circular cubic $Q_1q_2 - Q_2q_1 = 0$, or explicitly:

$$(x^2 + y^2)(y_1x - x_1y) + 2xy - y_1x - x_1y = 0. \tag{23}$$

This cubic is also generated by all pairs of corresponding points on the rays of the pencil through (x_1, y_1). Hence the theorem:

The locus of all pairs of corresponding points of a circular transformation on the rays of a pencil in a general position is a circular cubic.

The tangents from

$$P'\left(x_1' = \frac{x_1}{x_1^2 + y_1^2}, \qquad y_1' = \frac{-y_1}{x_1^2 + y_1^2}\right)$$

to the cubic pass through B_3, B_1, B_2, i. e., they have these as points of tangency. As B_1 and B_2 are the circular points at infinity, P' is the real focus or center of the cubic. The fourth (real) tangent passes through P as the point of tangency. The direction of the real asymptoteis determined by the slope y_1/x_1 or the ray through P and B_3. The equation of the asymptote is therefore $y = (y_1/x_1)x + k$, in which k is easily determined as $k = 2y_1/(x_1^2 + y_1^2)$. The asymptote cuts the cubic in the principal point H with the abscissa $h = x_1(k^3 + k)/2(k - k^2y_1 - y_1)$ and the ordinate $j = (y_1/x_1)\,h + k$.

The properties of the inversion $z' = 1/z$ lead to the following results: Every pair of corresponding or inverse points on the cubic, like A and A' on a ray through P is concyclic with the double points A_1 and A_2, Fig. 3; in fact $(AA'A_1A_2) = -1$. The same pair of points also lies on a circle of the orthogonal conjugate pencil of circles. The same is true for any other pair of inverse points on the cubic. Hence the theorem:

The circular cubic (elliptic serpentine) may be produced by the pencil of circles through A_1A_2 and a projective conjugate pencil.

Designating the first pencil of circles by (S_1), the second by (S_2) and the corresponding pencil of rays through P by (T), the theorem follows: *The same cubic may be produced by either pencil* (S_1) *or* (S_2) *and the corresponding projective pencil* (T).

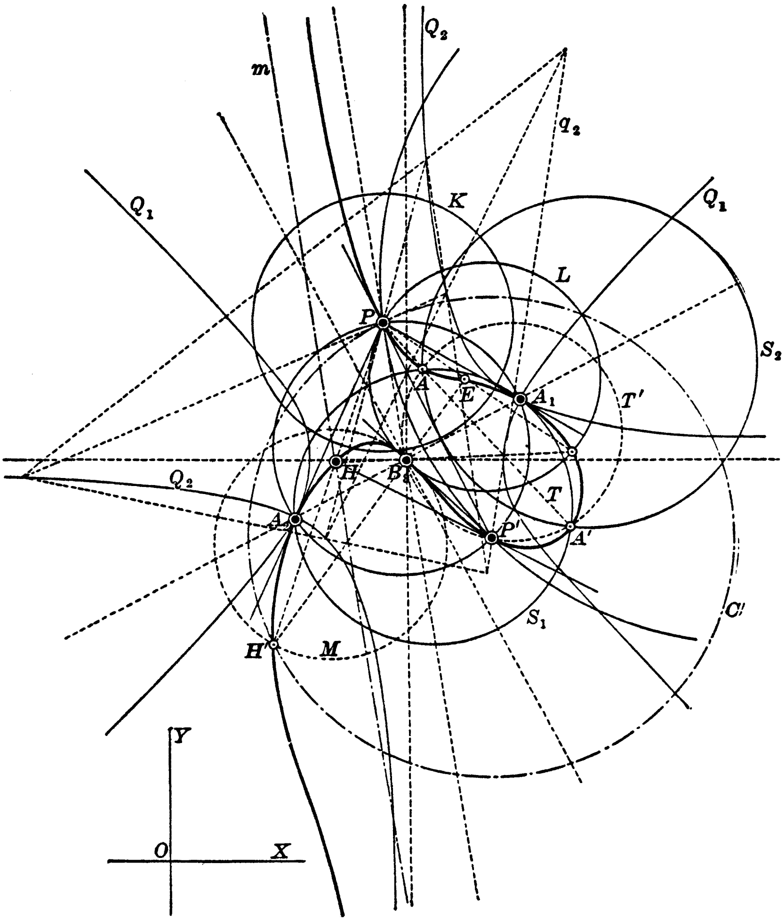

FIG. 3.

If S_1, S_2, T is a corresponding set in this projectivity, the tangents from P to S_1 and S_2 are equal, since $PA \cdot PA'$ equals the squares of these tangents. S_1 and S_2 are therefore orthogonal to a circle K with P as a center. Conversely any pair of (S_1) and (S_2) simultaneously orthogonal to a circle of the concentric pencil around P cuts out two inverse points on the cubic. Hence the theorem:

If for every circle of a concentric pencil an orthogonal pair (orthogonal to the given circle) from two conjugate pencils of circles is determined, then the locus of the points of intersection of all such pairs is a circular cubic.

By means of this theorem it is easy to construct any number of points of the cubic. To get such a pair of points draw any circle S_1 of the first pencil; next draw the circle K with P as a center and orthogonal to S_1; then draw a circle of (S_2) (center on A_1A_2 produced) orthogonal to S_1 and K. The intersections A and A' of S_1 and S_2 are two required points.

By the inversion $z' = 1/z$ the pencils (S_1) and (S_2) are transformed into themselves, while the pencil of rays (T) is transformed into the pencil of circles through B_3 and P', (T'), and there is clearly

$$(T') \barwedge (S_1) \barwedge (S_2).$$

The cubic is therefore also produced by the following projectivities: (T') and (T), (T') and (S_1), (T') and (S_2). It is known that every conic of a pencil of conics through four points of a cubic cuts the cubic in two other points and that all rays joining two such points concur in a point of the cubic. Hence, every tangent circle to the cubic at B_3 cuts the cubic in two points and the ray joining them passes through a fixed point E of the cubic. In the figure this point was determined by means of the tangent-circle L. Conversely, every ray through E cuts the cubic in two points which with B_3 determine a tangent-circle at B_3. Hence, connecting E with B_3 and producing, the cubic is cut in a point H', so that three points of the cubic and on the corresponding tangent circle are now assembled at B_3. This circle, M with B_3H' as a diameter, is therefore osculating the cubic at B_3, and its inverse in the real asymptote m of the cubic. The inverse H of H', through which m passes is called the principal point of the cubic.

The concentric pencil of circles around P' touches the cubic at the circular points. The rays connecting every pair of points of intersection of circles of this pencil with the cubic concur in a point of the cubic. In the limiting case of an infinitely large circle (with P' as a center), the ray joining the two points of intersection is the asymptote. Hence, H is the point of concurrence of those rays, and the well-known theorems follow:

Every ray through the principal point H of the circular cubic cuts the latter in two points which are equidistant from the center (real focus) P'.

The circular cubic is the product of a pencil of rays and a projective pencil of concentric circles.

An interesting particular case is obtained when the point P, Fig. 3, is assumed infinitely distant. In this case the circular points at infinity are conjugate with respect to the cubic and the points P' and H coincide with B_3, Fig. 4. The curve is symmetric with respect to B_3 and the tangents at A_1 and A_2 are parallel to the asymptote m. The construction of the curve itself is remarkably simple: Draw any line perpendicular to m and cutting the coördinate-axes at σ_1 and σ_2; $\sigma_1\sigma_2$ corresponds to one of the circles

K of the concentric pencil around P in the general case. With σ_1 as a center and $\sigma_1 A_2$ as a radius draw a circle S_1; from σ_2 draw tangents to S_1; the points of tangency of these, A and A', are two corresponding points of the cubic and are on a ray $T \parallel m$. A and A' are, of course, also situated on the circle S_2 of the conjugate pencil to (S_1), and on a circle T' tangent to the cubic at B_3. B_3 is a point of inflexion of the cubic. Designating the slope of m by μ, the equation of the cubic, according to (23), becomes

$$(x^2 + y^2)(\mu x - y) - \mu x - y = 0. \tag{24}$$

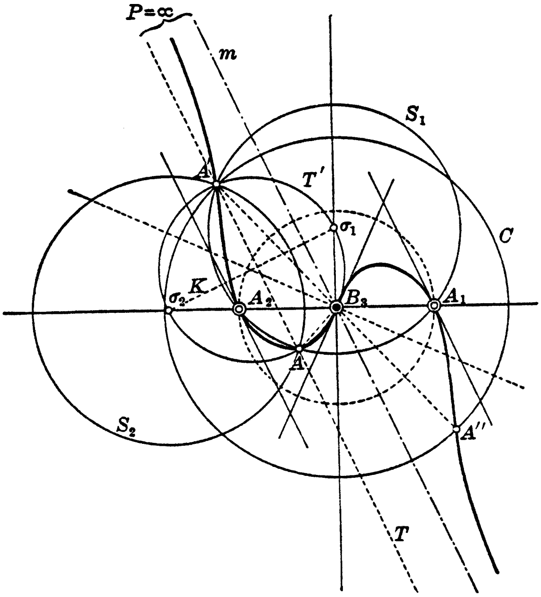

FIG. 4.

5. The Invariant Nets.

It has been seen that with every point P in a Steinerian transformation is associated an invariant cubic. If the transformation is at the same time circular, the cubic also will be circular and of the elliptic serpentine type. In an involutoric circular transformation, there exists therefore a net of such cubics, of which every individual remains invariant. *The net is produced by all the projectivities that may be established between the two conjugate pencils of circles having the double points of the transformation as a base.* Two circles chosen at random from the two pencils determine a transformation, which clearly is possible in a doubly infinite number of ways. The equation of the net belonging to the transformation $z' = 1/z$ is

$$(x^2 + y^2)(y_1 x - x_1 y) + 2xy - y_1 x - x_1 y = 0. \tag{25}$$

These transformations of the cubic in itself, or central correspondences, are of the first kind. There are ∞^4 such transformations and consequently that many nets and ∞^6 circular elliptic serpentines admitting of such transformations.

The second kind of involutoric circular transformations of a cubic in itself, the noncentral correspondences on the cubic, are obtained in the following manner. In Fig. 5 let O be the origin; on the real axis

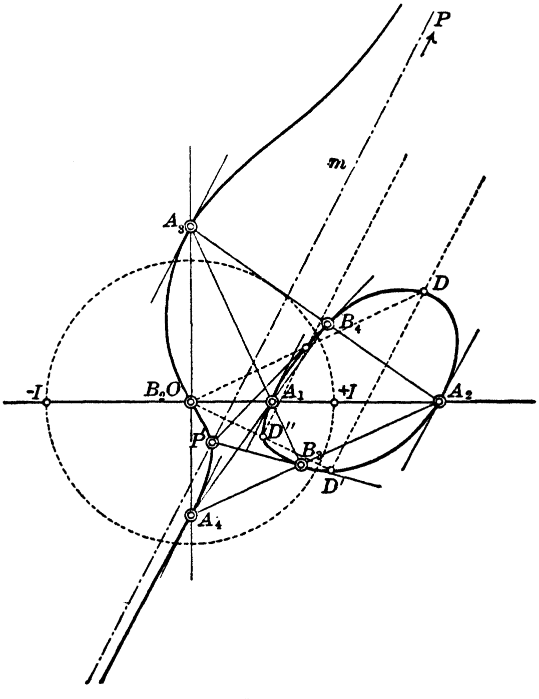

FIG. 5.

$OA_1 \cdot OA_2 = +1$; and on the imaginary axis $OA_3 \cdot OA_4 = -1$. Draw the diagonal points $B_2 (\equiv 0)$, B_3, B_4, and construct a cubic associated with an infinitely distant point P, in the Steinerian transformation belonging to the quadruple $A_1A_2A_3A_4$. Such a cubic passes through the circular points at infinity, hence through nine points (the two circular points, the A's and B's) which as a whole remain invariant in the inversion $z' = 1/z$. Designating by $P = 0$ and $Q = 0$ the equations of any two conics through the A's and by $p = 0$ and $q = 0$ the polars of $P\infty$ with respect to $P = 0$ and $Q = 0$, the equation of the cubic will be $Pq - Qp = 0$. If the coördinates of A_i are x_i, y_i, we may assume for P and Q the expressions

$$P \equiv xy \qquad \text{and} \qquad Q \equiv (xx_1y_3 + y - y_3)(x - yx_1y_3 - x_1),$$

so that the polars of $P\infty$ with respect to P and Q are represented by

$$p \equiv y + mx,$$

and

$$q \equiv [x_1y_3 + \frac{m}{2}(1 - x_1^2y_3^2)]x + [\tfrac{1}{2}(1 - x_1^2y_3^2) - mx_1y_3]y - \tfrac{1}{2}[x_1^2y_3 + y_3 + m(x_1 - x_1y_3^2)].$$

The equation of the cubic in its explicit form is now

$$(26) \quad (y - mx)(x^2 + y^2) + \frac{m(1 - x_1^2)}{x_1}x^2 + \frac{1 - y_3^2}{y_3}y^2 - mx - ey = 0.$$

For every set of values x_1, y_3, m an invariant cubic is obtained. With every inversion $z' = 1/z$ a triply-infinite system of invariant cubics may thus be established. For all these cubics the transformation is non-central, or of the second kind, and is one of the three which are involutoric and of the second kind for all circular cubics through the fundamental quadruple $A_1A_2A_3A_4$ and its diagonal points. The other two have their centers at B_3 and B_4. Geometrically the relation between the transformations J and E of the first and second kind is as stated at the beginning. By a J with P as a center a point D is transformed to D'; by a J with O as a center D' is transformed to D''. By an E with O as a center, D is transformed to D''. Thus, the product of two J's is an E.

As there are ∞^4 circular involutoric transformations in a plane, there are as many systems of invariant cubics, or ∞^7 circular cubics bearing non central involutoric correspondences. As the general equation of a circular cubic depends on 7 essential constants it follows, that by means of Steinerian transformations and their circular specialization it is possible to derive all circular cubics of a plane and their properties.

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